Coloring in Graph Streams via Deterministic and Adversarially Robust Algorithms

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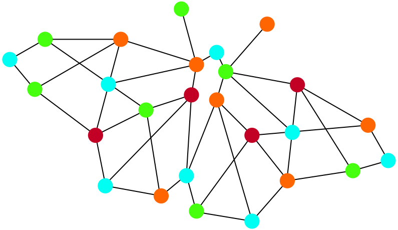

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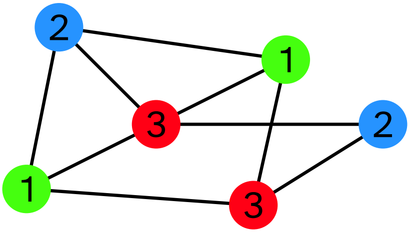
Δ-based coloring

- Graph $G = (V, E)$ with n vertices and max degree Δ
- Vertex coloring: assign a color to each vertex G so that no edge connects two vertices of the same color



- Finding vertex colorings with a *minimum* number of colors is NP-hard
- Δ-*based colorings* use number of colors depending on Δ.
  - Greedy algorithm: $\Delta + 1$ colors
  - Linial's algorithm[Linial92]: $O\left(\Delta^2\right)$ colors
  - Brook's theorem: Δ colors (if possible)

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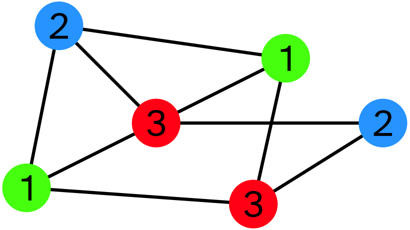

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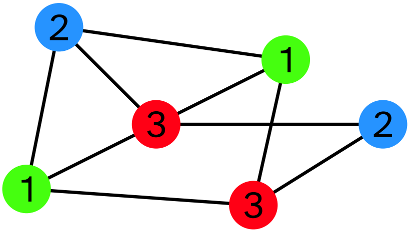

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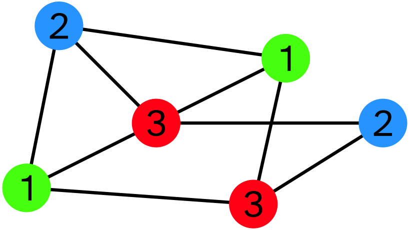

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To what extent is randomization necessary for streaming algorithms that compute a Δ-based coloring?

Outline

Deterministic multi-pass $(\Delta + 1)$ coloring

Adversarially robust coloring with $O\left(\Delta^{2.5}\right)$ colors

Deterministic multi-pass $(\Delta + 1)$-coloring on a graph stream

Input:

- A graph $G = (V, E)$ on n vertices with maximum degree Δ, provided as a sequence of edges

Processing:

- Limited working space: only "*semi-streaming*" ($\tilde{O}(n)$, where $\tilde{O}(\cdot)$ hides polylog factors in n and Δ.)[†]
- For each pass, algorithm reads the input edge sequence in order

Output:

- A coloring $\chi : V \to [\Delta + 1]$, so that if $\{u, v\} \in E$, then $\chi(u) \neq \chi(v)$

[†] Storing G takes $\tilde{O}(n\Delta)$ space.

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- [AssadiChenKhanna19] Single-pass randomized streaming algorithm for $(\Delta + 1)$ coloring, using semi-streaming space
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Is there a multi-pass deterministic semi-streaming space algorithm for $\Delta + 1$ coloring?

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Theorem

Same bounds hold for (degree+1) list coloring (D1LC), where each vertex $x \in V$ has associated list L_x of permitted colors, where $|L_x| \geq \deg x + 1$.

- Issue: storing color lists would take up to $\tilde{\Theta}(n\Delta)$ space. See paper.

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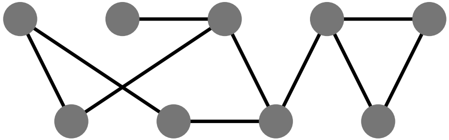
- Will repeatedly fix colors for more vertices
- Propose colors for all unfixed vertices, with few monochromatic edges $\implies$ can fix a constant fraction of proposed colors
- Example:



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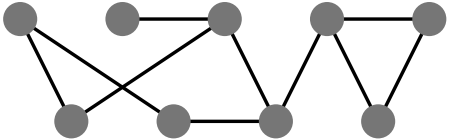

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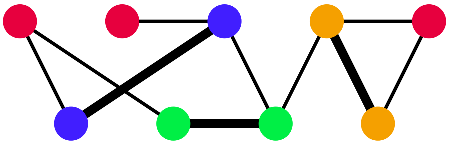

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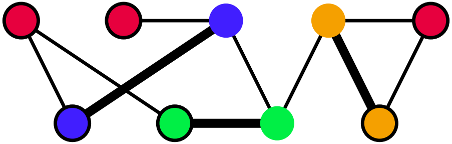

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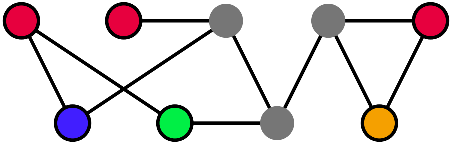

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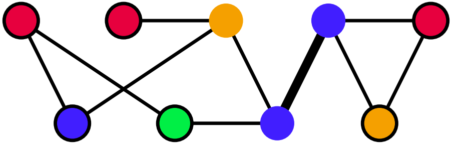

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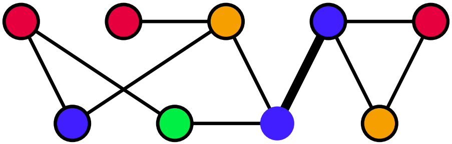

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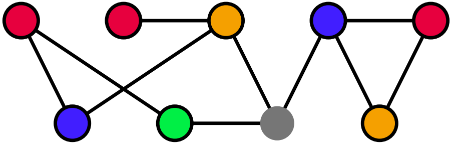

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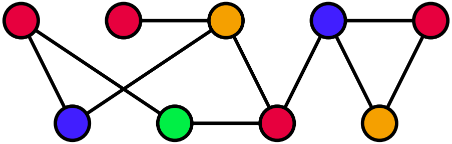

Deterministic color proposal

Progressively restrict which colors each vertex may have.

- Assign each uncolored vertex set $[\Delta + 1]$ of all colors
- Repeatedly choose a subset of the current color set for each vertex
  - Have a cost function[‡] bounding the final number of monochromatic edges
  - Pass 1: Compute "slack" values[§], where if x has color set S, then

$$\text{slack}(x, S) = |S| - |\{\{y, x\} \in E : y\text{'s color fixed and in } S\}|$$

  - Pass 2-3: Use hash family to search for good color subset assignment
- After $O(\log \Delta)$ refinements, have a single proposed color for every vertex.

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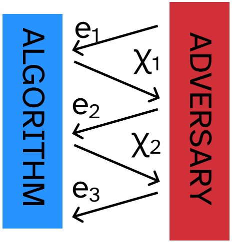
- Two player game between Algorithm and Adversary
- Adversary constructs series of inputs $e_1, e_2, \ldots e_i$, and Algorithm produces outputs $\chi_1, \ldots, \chi_i$ solving task for stream up to this point.
- Adversary's chosen inputs may depend on prior outputs of the Algorithm.
- Algorithm is "adversarially robust" if it has low error rate against *any* Adversary strategy
  - Example: Input generated in real time – outputs may influence future inputs
  - Sub-component of larger algorithm



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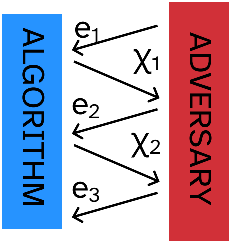

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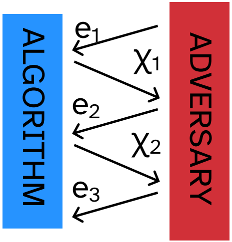

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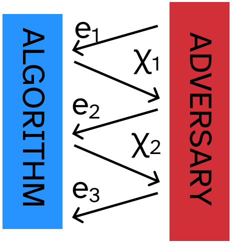

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- Space/color tradeoff: for any $\beta \in [0,1]$, get $\tilde{O}\left(\Delta^{5/2-3\beta/2}\right)$ colors with $\tilde{O}\left(n\Delta^{\beta}\right)$ space and $\tilde{O}\left(n\Delta\right)$ random bits.

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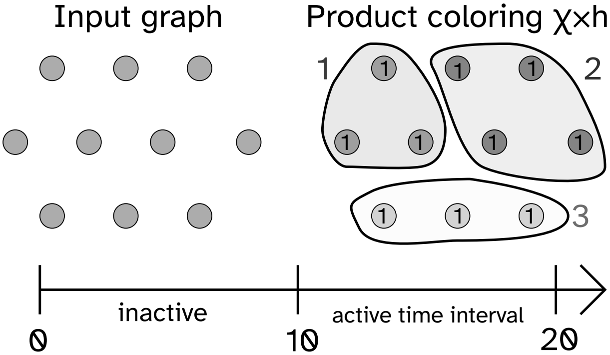
Example: Robust product coloring

- Use random partitions to reduce edges stored / increase colors used
- Adversarial robustness: periodically change partitions to avoid storing many edges

- Random partition $h : V \to [k]$
- Store edge $\{a, b\}$ into set D if $h(a) = h(b)$
- Compute coloring χ of D, and color v with $(h(v), \chi(v))$
- *Before* h is active, $|D|$ is small
- *While* h is active, $|D|$ can grow quickly
- *After* h is active, discard D



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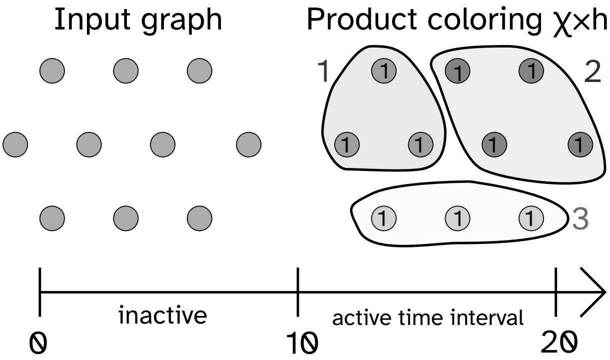

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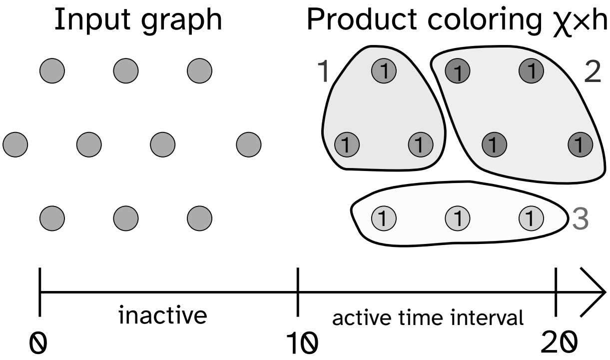

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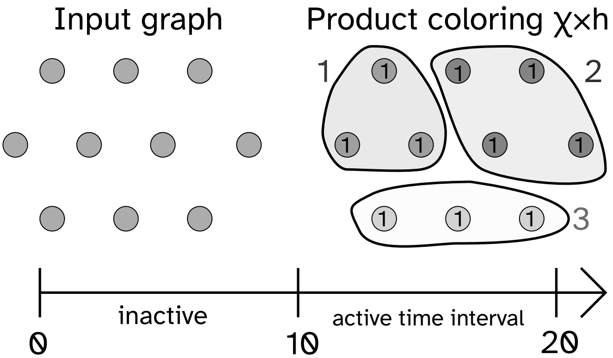

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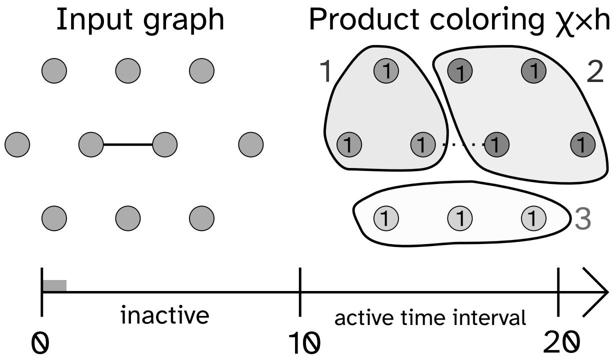

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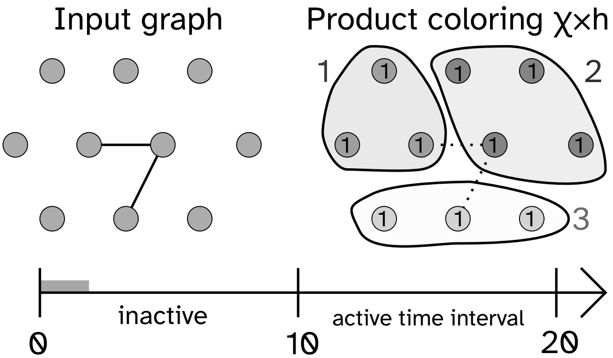

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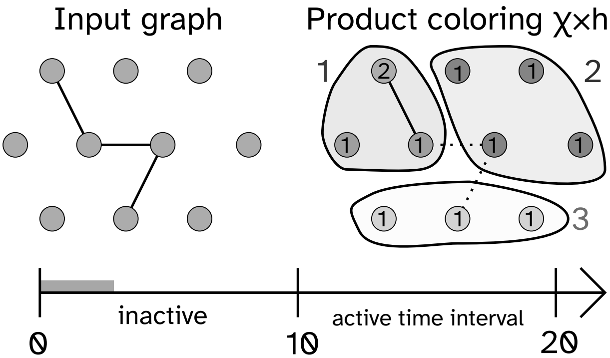

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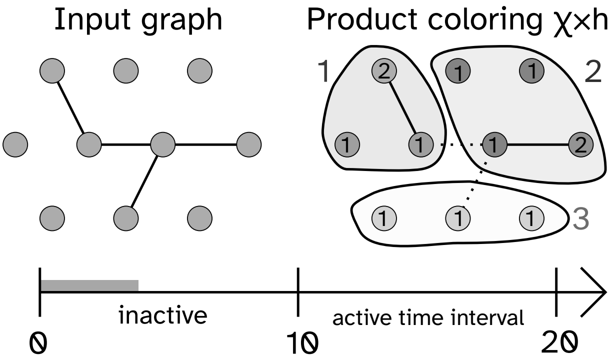

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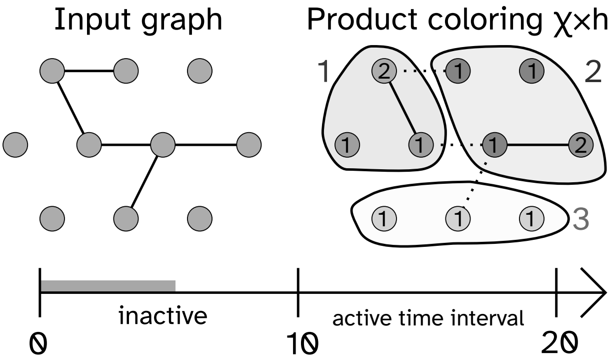

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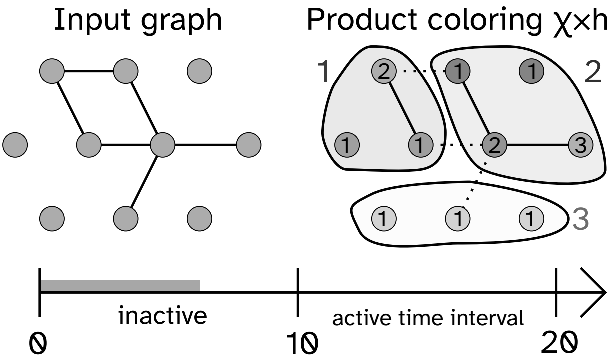

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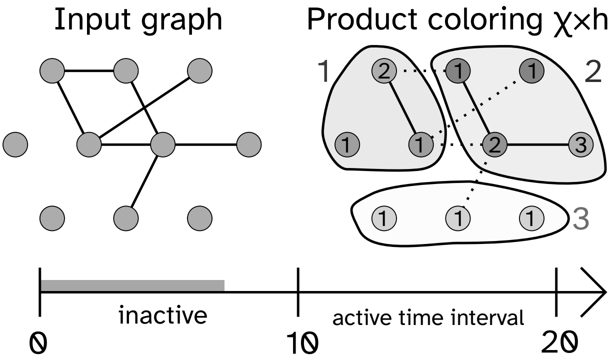

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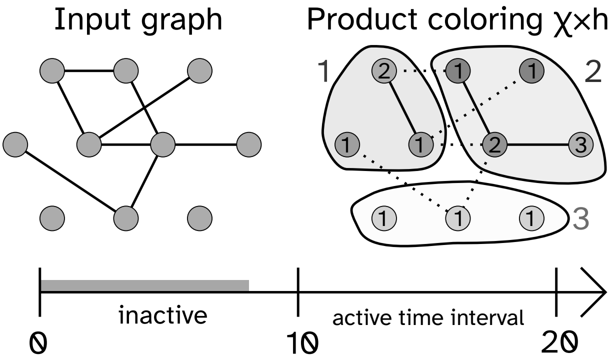

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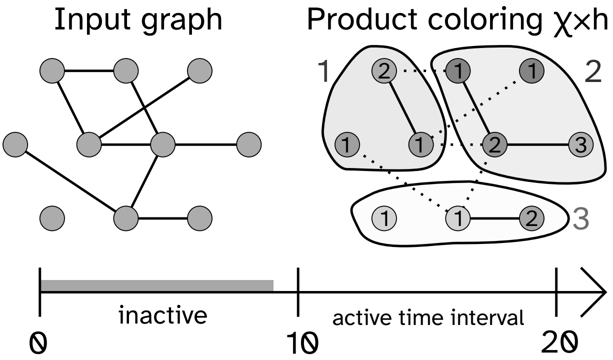

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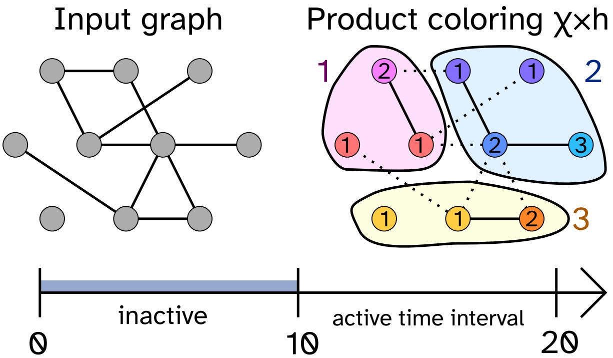

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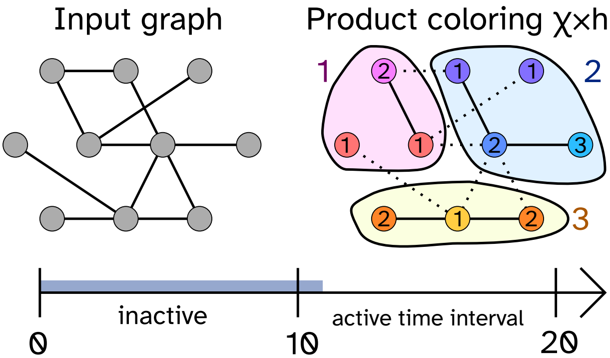

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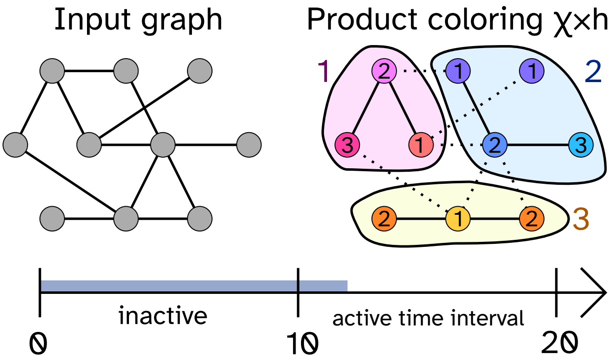

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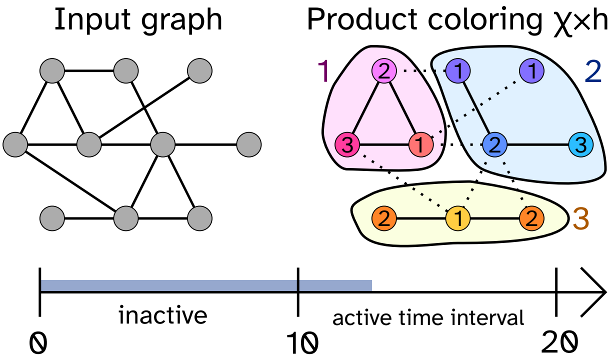

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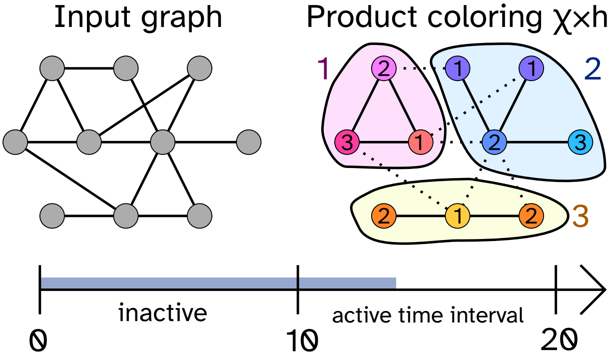

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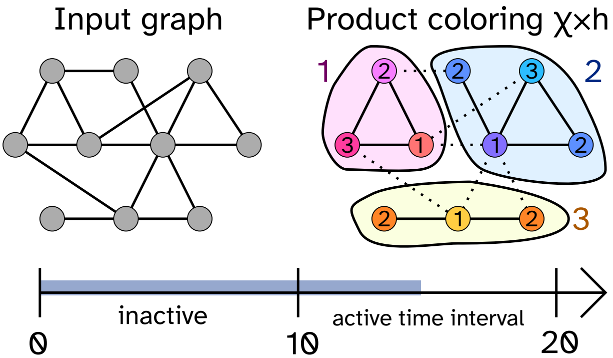

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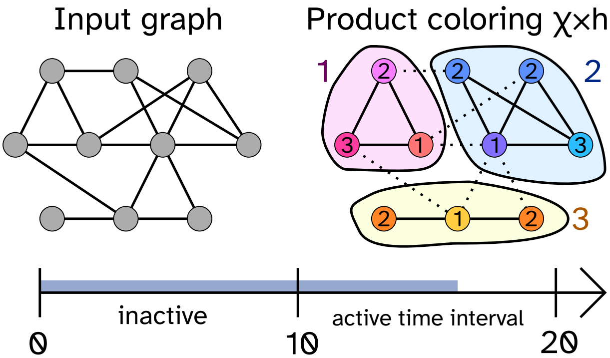

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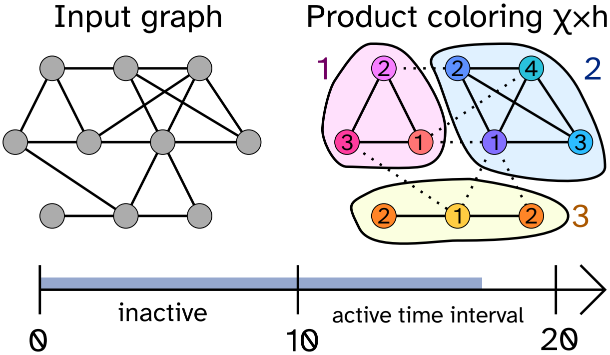

High level description of $O\left(\Delta^{2.5}\right)$-coloring algorithm

Different forms of product coloring algorithm are efficient for different edge insertion patterns.

"Slow" vertices

- $\leq \sqrt{\Delta}$ incident edges in a batch of n
- Time-linked product coloring instances

"Fast" vertices

- $> \sqrt{\Delta}$ incident edges in a batch of n
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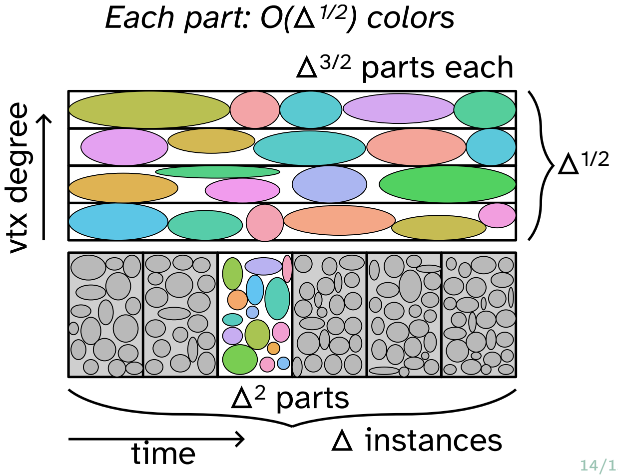

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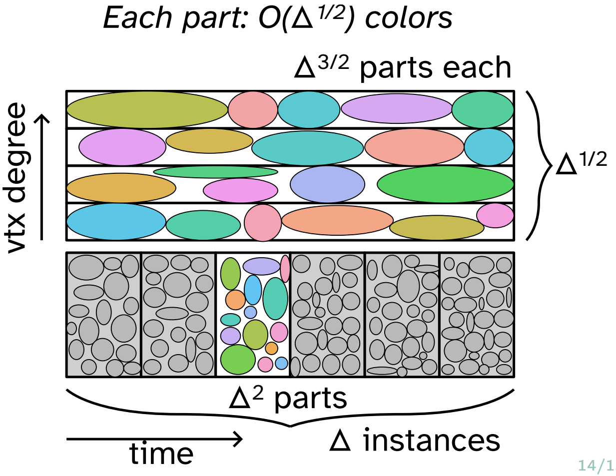

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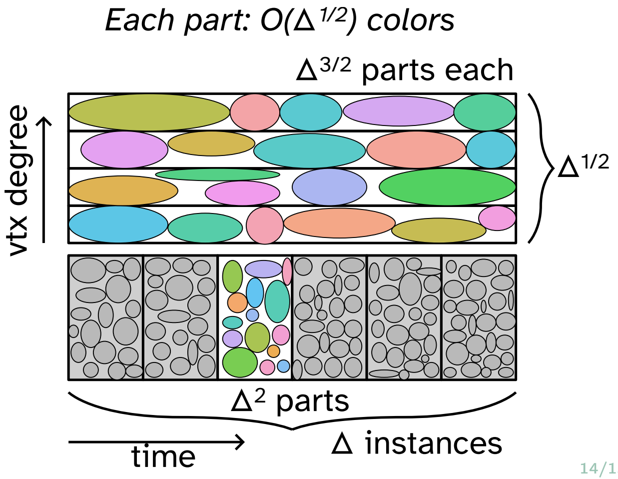

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