Finding Missing Items Requires Strong Form of Randomness

A. Chakrabarti[1] **M. Stoeckl**

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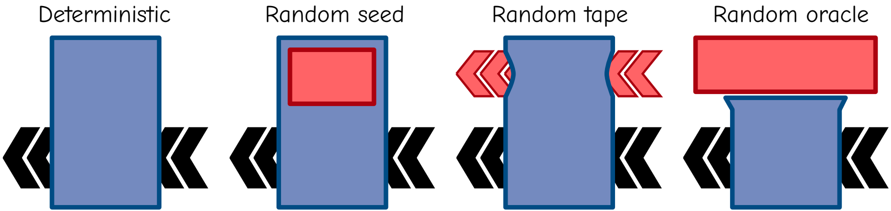

[1]This work was supported in part by the National Science Foundation under award 2006589.

Outline

Problem and models

- Missing Item Finding
- Adversarial setting for streaming algorithms
- Types of randomness for streaming algorithms

Our Results/Contribution

- Separations
- Random tape algorithm
- Random tape lower bound
- Pseudo-deterministic lower bound

Open problems

Goal

For streaming algorithms in the adversarial setting, are there significant separations in space complexity for the *different ways randomness can be used?*

Yes: Missing Item Finding

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Yes: Missing Item Finding

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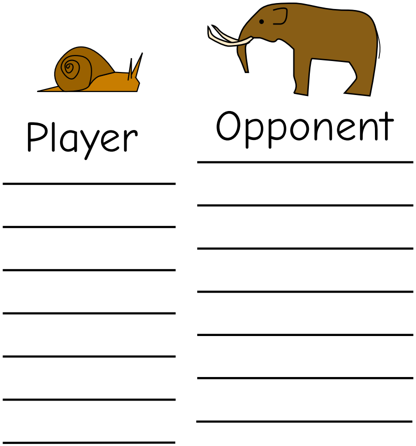
The Missing Item Finding Problem

Definition 1.
$\mathrm{MIF}(n, \ell)$ for $1 \leq \ell < n$

- ℓ-step game:
  - Receive: $a_i \in [n]$
  - Output: $o_i \in [n] \setminus \{a_1, \ldots a_i\}$
- Player is memory limited; opponent is not



Example 2 (On right).
$MIF(n = 10, \ell = 6)$

Two player game $\equiv$ Streaming algorithm with adaptive adversary

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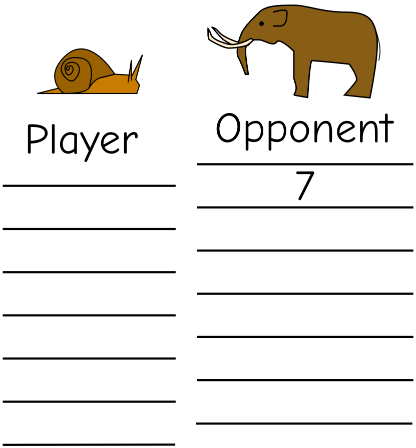

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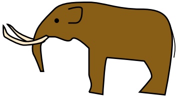

Player

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Opponent

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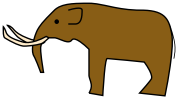

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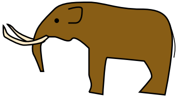

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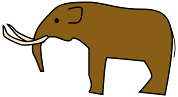

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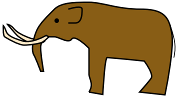

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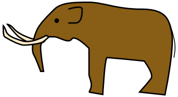

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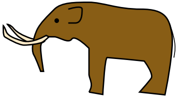

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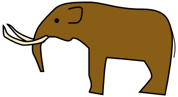

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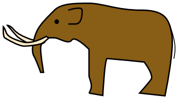

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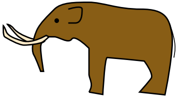

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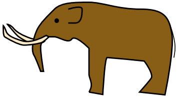

Player	Opponent
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Streaming algorithms

- **Streaming Algorithm:**
  - limited memory
  - processes sequence of elements one by one

Algorithm for fixed MIF(n, ℓ) input streams with $\ell \ll n$

```
S ← random subset of [n] of size O(1)
for e from input stream
    if e ∈ S:
        remove e from S
    report: arbitrary element of S
```

Performance requirements

- This talk: state machine view (ignores description/computational cost).
- Tracking error: algorithm makes output after every input, correct iff all outputs are
- Algorithm has cost s if worst-case memory usage is s bits
- An algorithm has error $\leq \delta$ in the:
  - **static setting**: if it has error probability $\leq \delta$ for any input stream
  - **adversarial setting**:[2] if it has error probability $\leq \delta$ for *any adaptive adversary*

static setting $\rightarrow$ **classic algorithm** ; adversarial setting $\rightarrow$ **robust algorithm**

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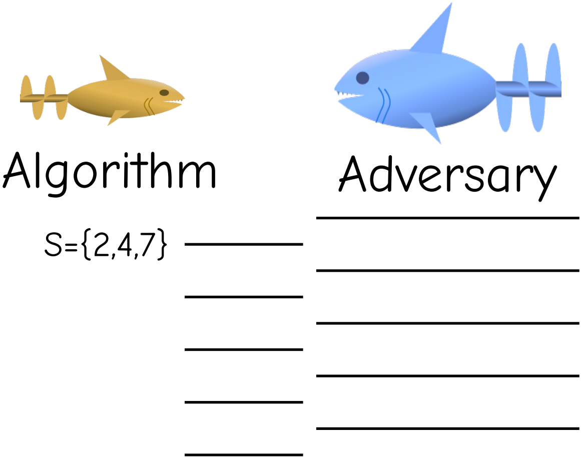
The adaptive adversary

- Can choose next input element as function of preceding outputs
- This correlates input with private algorithm randomness



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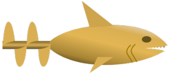
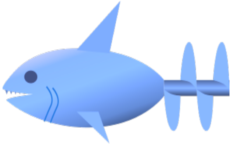


Algorithm



Adversary

S={2,4,7}

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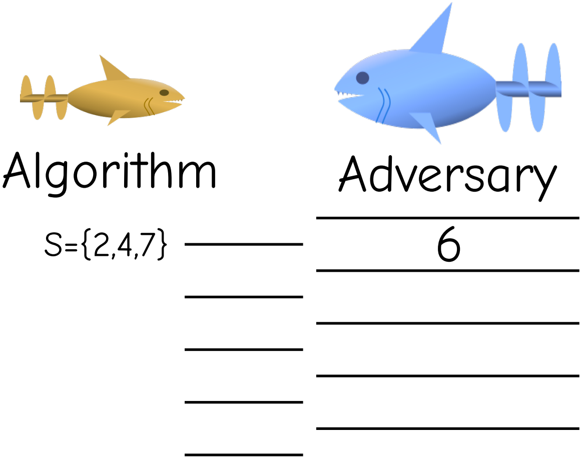

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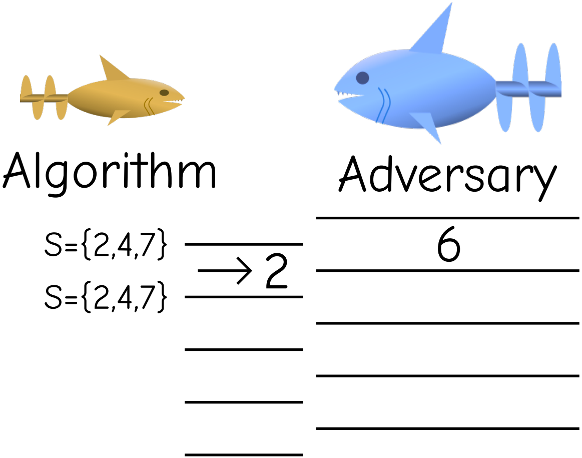

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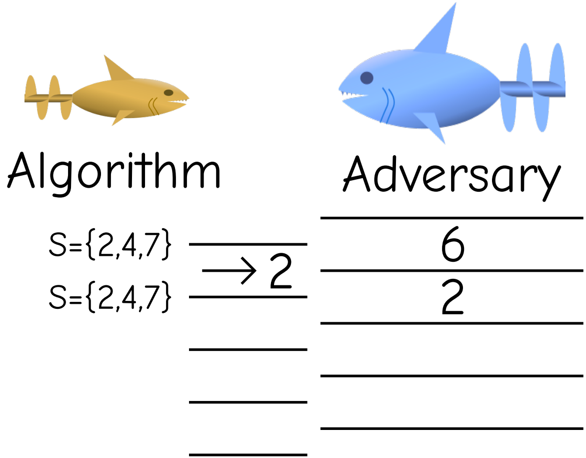

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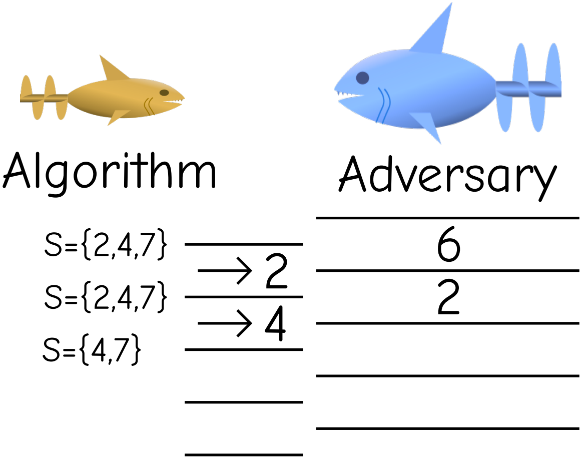

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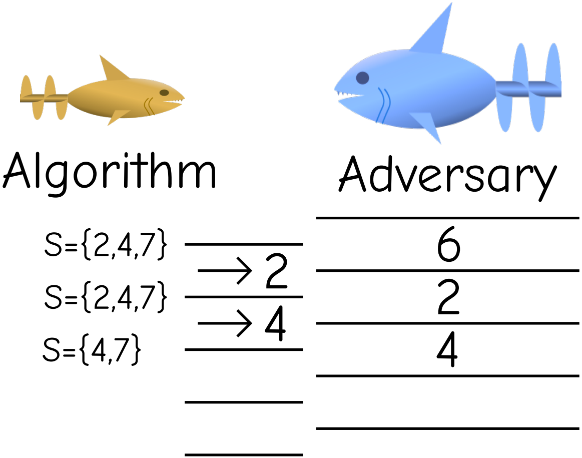

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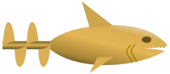

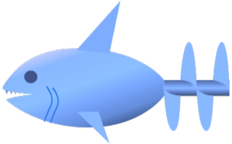

Algorithm		Adversary
S={2,4,7}		6
	→ 2	
S={2,4,7}		2
	→ 4	
S={4,7}		4
	→ 7	
S={7}		

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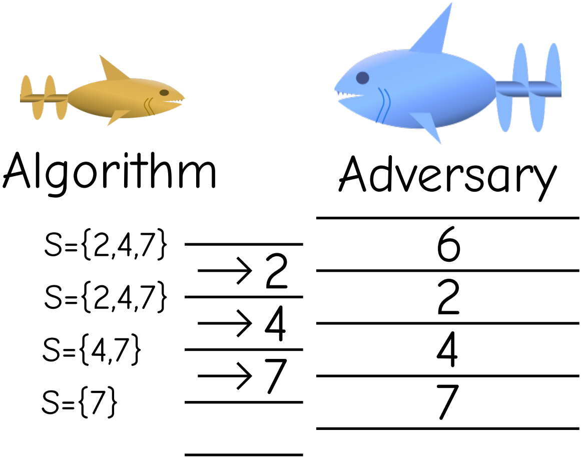

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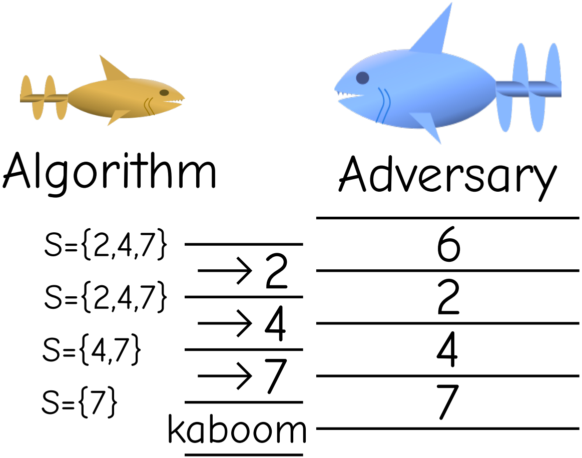

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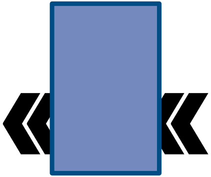
Four natural categories of streaming algorithms, by randomness type

Deterministic



No randomness

- Exact counter
- Sparse recovery
- Greedy matching

Random seed

Initial state random

- Linear sketch with random hash function
- Rabin fingerprint
- Example MIF algo

Random tape

Transitions random

- Reservoir sampling
- Morris Counter
- Noise / Differential privacy

Random oracle

Free access to long, persistent, random string

- Linear sketch with i.i.d. entries

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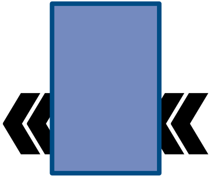

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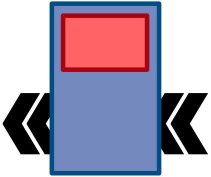

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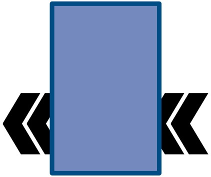

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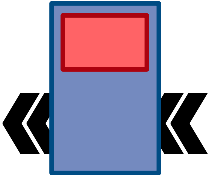

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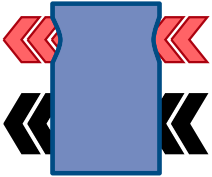

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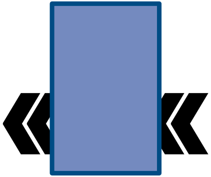

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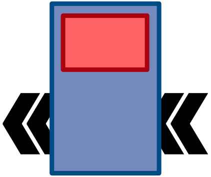

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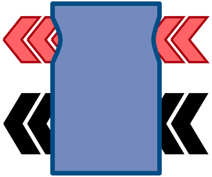

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Randomness type (almost) does not matter in static setting or with bounded adversaries

Emulate random oracle or random tape algorithm using random seed

Newman's theorem[a]

- Immediate corollary: any ϵ-error random oracle streaming algorithm with Q possible inputs has random seed emulation with $\epsilon(1+\delta)$ error and $+O\left(\log \frac{\log Q}{\epsilon\delta}\right)$ bits of space
- Non-constructive
- # adversaries $= \exp(\#$ streams$)$

[a] Ilan Newman. Private vs. common random bits in communication complexity. *Inform. Process. Lett.*, 39(2):67–71, 1991

Pseudo-random generators:

- If one-way functions exist and adversary is poly-time, ...
- If adversary has less memory than algorithm ...
  - Nisan's PRG[a]

[a] Noam Nisan. Pseudorandom generators for space-bounded computation. In *Proc. 22nd Annual ACM Symposium on the Theory of Computing*, pages 204–212, 1990

Randomness type (almost) does not matter in static setting or with bounded adversaries

Emulate random oracle or random tape algorithm using random seed

Newman's theorem[a]

- Immediate corollary: any ϵ-error random oracle streaming algorithm with Q possible inputs has random seed emulation with $\epsilon(1+\delta)$ error and $+O\left(\log \frac{\log Q}{\epsilon\delta}\right)$ bits of space
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Outline

Main Result

For streaming algorithms in the adversarial setting, are there significant separations in space complexity for different ways randomness can be used?

MIF (n, ℓ) space, adversarial setting

	$\ell = 2^{\sqrt{\log n}}$	$\ell = \sqrt{n}$
Random oracle	$\tilde{\Theta}(1)$	$\tilde{\Theta}(1)$
Random tape	$\tilde{\Theta}(1)$	$\tilde{\Omega}(\ell^{1/4})$
Random seed	$\tilde{\Theta}(\sqrt{\ell})$	$\tilde{\Theta}(\sqrt{\ell})$
Deterministic	$\tilde{\Theta}(\ell)$	$\tilde{\Theta}(\ell)$

(Hiding polylog(n, ℓ) factors; at error $\delta = \frac{1}{n^2}$)

Yes: for RT/RO and RS/RT

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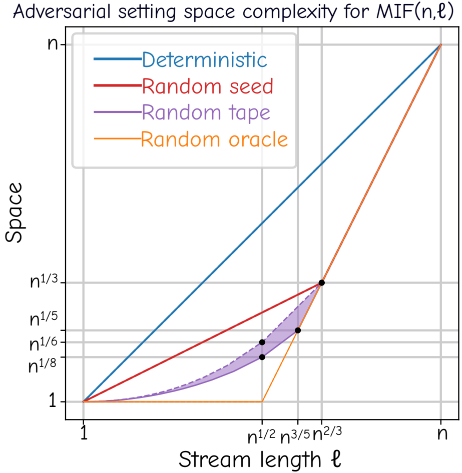


Specific results of this paper

1. Lower bound for random tape in adversarial setting
2. Upper bound for random tape in adversarial setting
3. Lower bound for *pseudo-deterministic* algorithms
4. Corollary via older work[a]: $\implies$ lower bound for random *seed* in adversarial setting

[a] Manuel Stoeckl. Streaming algorithms for the missing item finding problem. In *Proc. 34th Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 793–818, 2023. Full version at arXiv:2211.05170v1

Known results (this + Stoeckl 2023, 2024)

Setting	Type	Space
Static	Seed	$\tilde{\Theta}(1)$
Adversarial	Oracle	$\tilde{\Theta}(\ell^2/n + 1)$
Adversarial	Tape	$\Omega\left(\ell^{\frac{15}{32}\log_n \ell}\right)$ $\tilde{O}\left(\ell^{\log_n \ell}\right)$
Adversarial	Seed	$\tilde{\Theta}\left(\ell^2/n + \sqrt{\ell}\right)$
Pseudo-det.	Oracle	$\tilde{\Theta}(\ell)$
Any	Det.	$\tilde{\Theta}(\ell)$

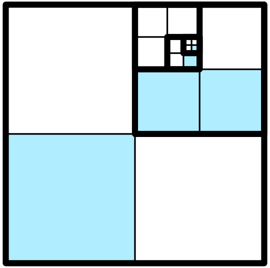
(Hiding polylog(n, ℓ) factors; at error $\delta = \frac{1}{n^2}$)

Other work: Magen 2024; Tarui 2007

Outline

Random tape algorithm (in adversarial setting): recursive structure

- Design: split universe $[n]$ into k blocks
- Run random-subset algorithm to choose a safe block
- Inside chosen safe block: run this algorithm on domain $[n/k]$



Init()

```
    Split [n] into B_1,...,B_k
    S ← random subset of [k]
    c ← S.pop()
    A ← recursive instance on B_c
```

Update(e)

```
    if e ∈ B_i for any i in S:
        remove i from S
    if e ∈ B_c:
        A.Update(e)
        if A is done:
            c ← S.pop()?
            A ← recursive instance on B_c
    report: A's output
```

Outline

Random tape lower bound (in adversarial setting): recursive structure

Lemma 3 (Stoeckl 2023).
Robust algorithms for MIF (n, ℓ) with $\leq \frac{3}{4}$ error probability require $\Omega\left(\ell^2/n\right)$ space

Lemma 4.
Lower bound for a z-bit robust random tape algorithm for MIF (n, ℓ) depends on the lower bound for MIF (w, t) with

$$w = \Theta\left(\frac{zn}{\ell}\right) \qquad t = \Theta\left(\frac{\ell}{z}\right)$$

Theorem 5.
Robust random tape algorithms for MIF (n, ℓ) require space:

$$\Omega\left(\max_k \left(\frac{\ell^{k+1}}{n}\right)^{\frac{2}{k^2+3k-2}}\right) = \Omega\left(\ell^{\frac{15}{32}\log_n \ell}\right)$$

Reduction step: searching for information must stop

1. Adversary sends $\ell/2$ random elements
   - Let ρ be resulting algorithm state
   - Not enough space to store all random elements: algorithm must overestimate, and only considers elements in a set H_ρ to be *safe*
   - Typically $|H_\rho| = O\left(\frac{zn}{\ell}\right)$
2. Adversary tries to identify H_ρ, in $\Theta(z)$ epochs
   - Have a possible set H_σ for each possible state σ; making an output outside H_σ is risky if $\sigma = \rho$
   - Each epoch, either:
     (a) There exists a "sub-adversary" for next $\Theta\left(\frac{\ell}{z}\right)$ steps which likely rules out half the remaining candidate H_σ values. If so, run it!
     (b) There exists a set W of size $O\left(\frac{zn}{\ell}\right)$ which probably contains *all* the next $\Theta\left(\frac{\ell}{z}\right)$ algorithm outputs, no matter what
   - Case (a) is unlikely to happen $\Theta(z)$ times – might end up ruling out H_ρ itself
   - Case (b): DONE – algorithm solves $MIF\left(O\left(\frac{zn}{\ell}\right), \Theta\left(\frac{\ell}{z}\right)\right)$ with inputs in W

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Outline

- Problem and models
  - Missing Item Finding
  - Adversarial setting for streaming algorithms
  - Types of randomness for streaming algorithms
- Our Results/Contribution
  - Separations
  - Random tape algorithm
  - Random tape lower bound
  - Pseudo-deterministic lower bound
- Open problems

What is pseudo-determinism?

- Randomized algorithm $\mathcal{A}$ that behaves like a deterministic one
  - There exists **canonical output function** $f_{\mathcal{A}}$ from inputs to outputs so that $\Pr\left[\mathcal{A}(x) = f_{\mathcal{A}}(x)\right] \geq 1 - \epsilon$ for all possible inputs x
- Pseudo-deterministic streaming algorithms:[3]
  - If correctness relation is a function, correct algorithms are pseudo-deterministic
  - Automatically work in the adversarial setting
  - Newman's theorem can apply

[3] For a paper introducing pseudo-determinism to streaming, see: Shafi Goldwasser, Ofer Grossman, Sidhanth Mohanty, and David P. Woodruff. Pseudo-Deterministic Streaming. In *Proc. 20th Conference on Innovations in Theoretical Computer Science*, volume 151, 79:1–79:25, 2020

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Pseudo-deterministic and random seed lower bounds

Theorem 6.

Pseudo-deterministic random-oracle algorithms for MIF (n, ℓ) with error $\delta = 1/poly(n)$ and $\ell = \Omega(\log n)$ require space

$$\Omega\left(\frac{\ell}{(\log n)^2}\right)$$

Theorem 7 (Stoeckl 2024).

A random seed streaming algorithm for adversarial setting with z bits of state processing a stream of length ℓ can be made to probably "behave pseudo-deterministically" for some contiguous stretch of $\Theta(\ell/z)$ inputs.

Corollary 8.

Random seed, adversarial setting, $\leq 1/6$ error, MIF (n, ℓ) algorithms require space[4]

$$\Omega\left(\frac{\ell^2}{n} + \sqrt{\frac{\ell}{(\log n)^3}}\right)$$

[4] The ℓ^2/n term comes from Lemma 3

Open problems

- Mirror Game: like MIF, but a) neither player can repeat numbers b) $n = 2\ell$ c) player starts[5]
  - Unknown: do space-efficient algorithms need a random oracle?
- Can we separate random *seed* and *tape* for adversarial setting turnstile L_0 estimation algorithms? Pseudo-deterministic gap hamming communication complexity still open.[6]

[5] Sumegha Garg and Jon Schneider. The Space Complexity of Mirror Games. In *Proc. 10th Conference on Innovations in Theoretical Computer Science*, 36:1–36:14, 2018, Feige 2019; Magen and Naor 2022; Menuhin and Naor 2022

[6] *Some progress:* Dmytro Gavinsky. Unambiguous parity-query complexity. *arXiv preprint arXiv:2401.11274*, 2024

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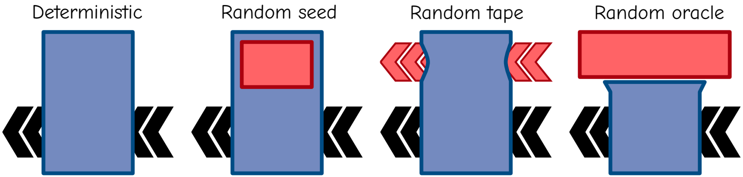
Conclusion



- Unlike in the static setting, the example of Missing Item Finding shows that space-efficient streaming algorithms in the adversarial setting may require a random tape or random oracle.

- Lower bound methods:
  - Random tape: Recursive structure of MIF algorithms + adversary iteratively searching for information on past states + a useful property when search cannot progress
  - Random seed: use semi-generic reduction to pseudo-deterministic
  - Pseudo-deterministic: generalize deterministic proof + alternate establishing canonical and actual algorithm properties

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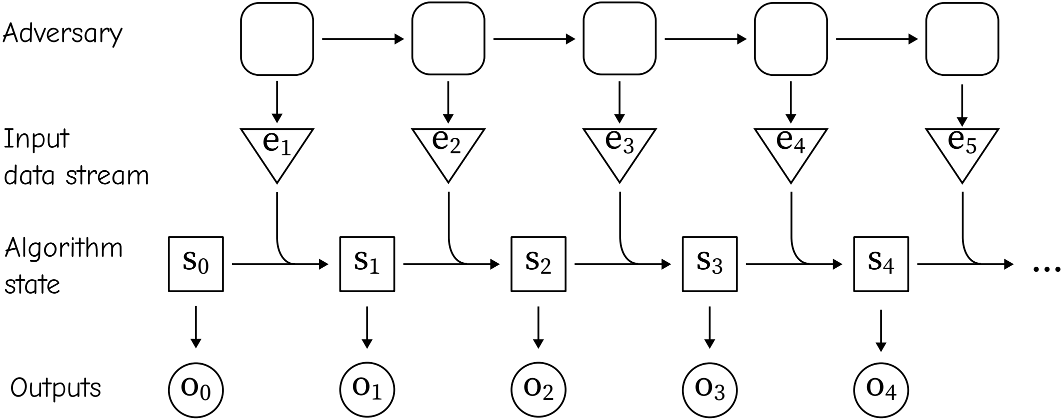
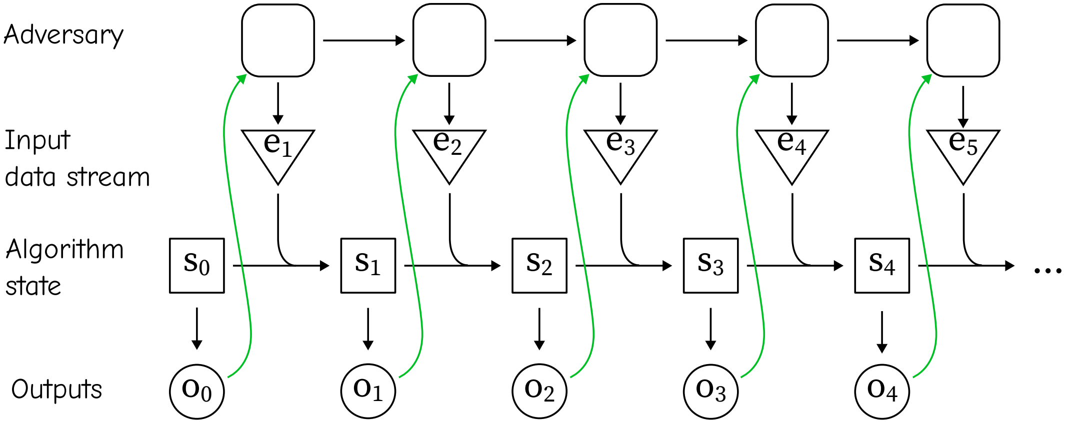
Bonus slides

1. State machine views (of randomness types, models.)
2. Random oracle algorithm explanation
3. Random-seed to pseudo-deterministic explanation
4. Full statements of main theorems
5. Reduction step for random tape lower bound
6. Simplified FindCommonOutputs
7. Related work

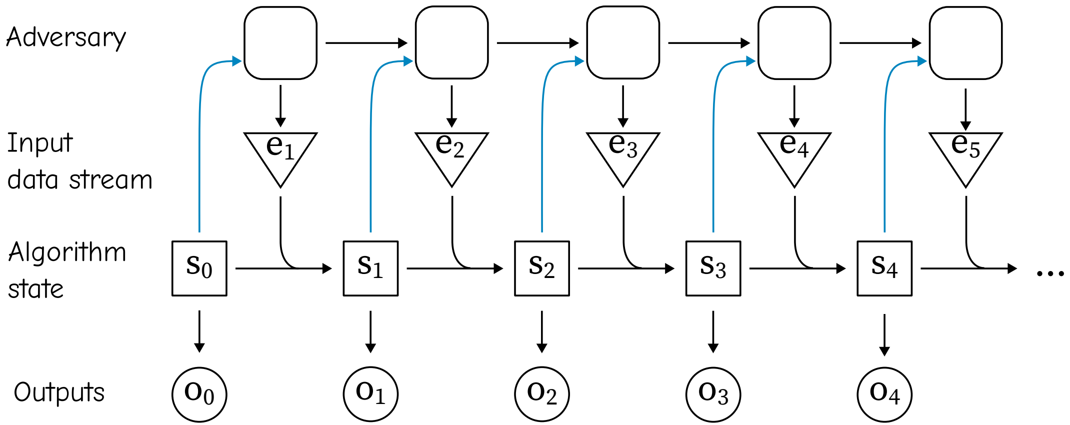
State machine perspective: static setting



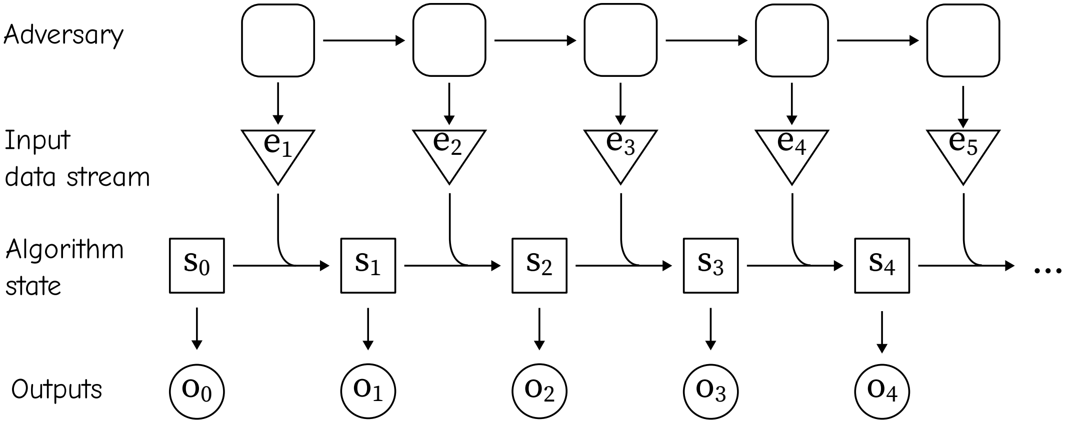
State machine perspective: adversarial setting



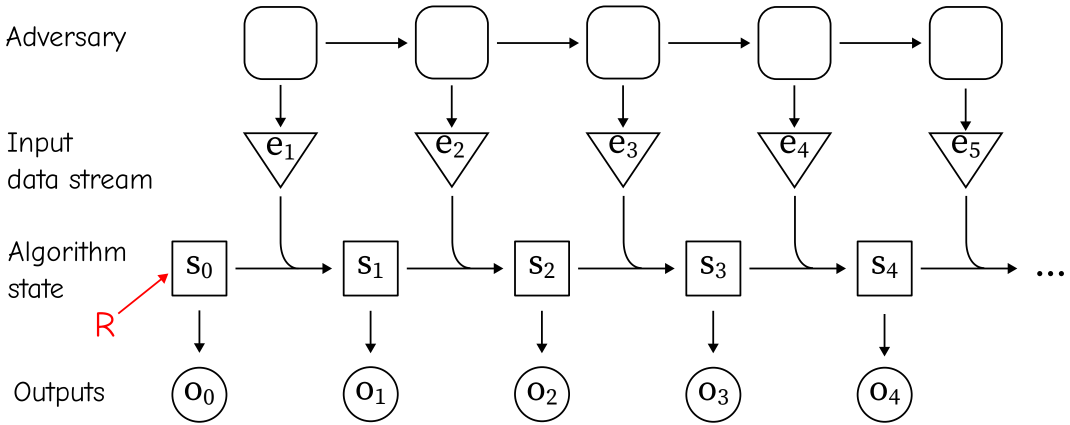
State machine perspective: white-box adversarial setting



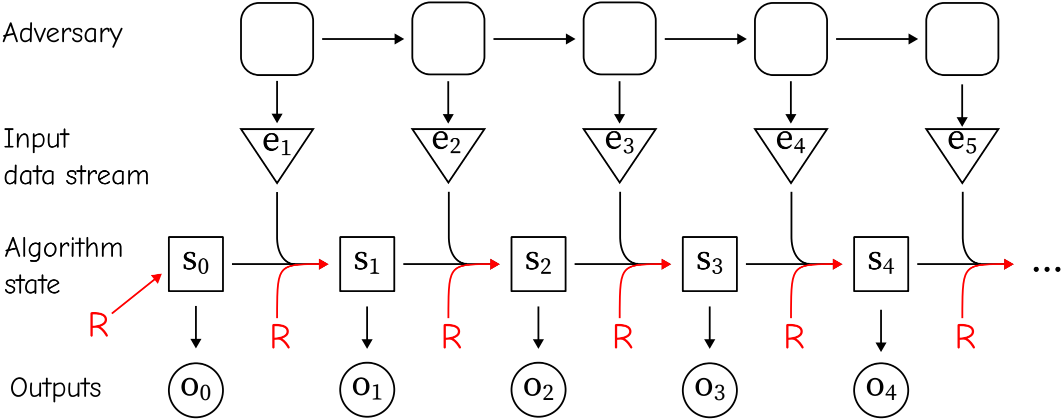
State machine perspective: deterministic



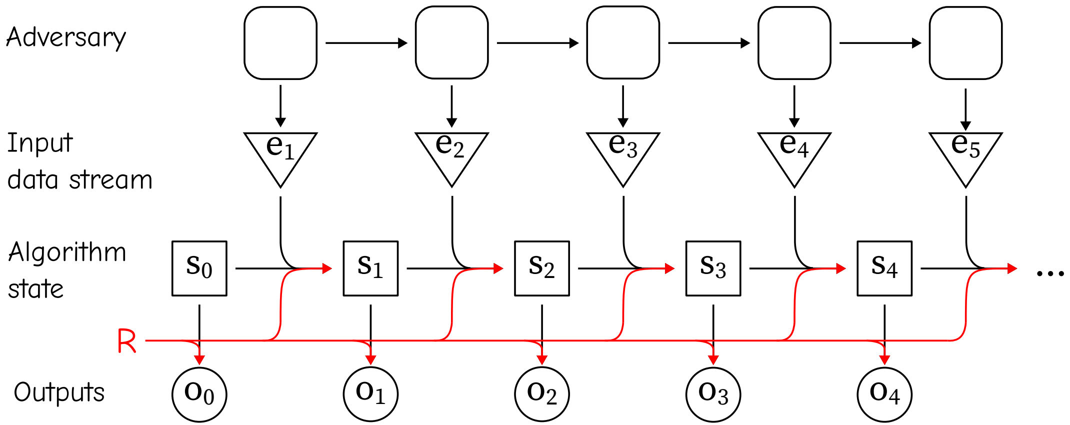
State machine perspective: random seed



State machine perspective: random tape

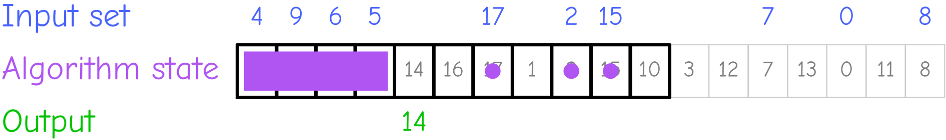


State machine perspective: random oracle



Random oracle algorithm

- In "random sample" approach, S can be drawn from oracle randomness
- If always outputting *least* available element of S, can efficiently encode removed elements as union of contiguous and sparse sets



Random seed to pseudo-deterministic

Consider adversary with $\Theta(z)$ epochs of length $t = \Theta(\ell/z)$.

For each epoch:

1. If $\exists$ subsequence x of length t for which, conditioned on history, algorithm output sequence has high entropy (≥ 0.5 bits, say):
   - Send x to algorithm. Next epoch.
2. Otherwise, for all possible x, conditional entropy of outputs is low (≤ 0.5 bits) which implies some particular output sequence occurs with probability $\geq \frac{2}{3}$.[7] This is pseudo-determinism.

Observe: entropy of random seed is limited by z, so case 1 can only occur $\geq 4z$ times, a constant fraction of the time.[8]

---

[7] Say all output sequences have $\leq \frac{2}{3}$ probability. Then there exists a subset S of possible outputs with net probability between $\frac{1}{3}$ and $\frac{2}{3}$; $H(X \in S) \geq -\frac{1}{3}\log\frac{1}{3} - \frac{2}{3}\log\frac{2}{3} \geq 0.798$.

[8] $H(S) \geq H(O_1) + H(O_2|O_1) + \ldots) \geq \frac{1}{2} \cdot \frac{1}{2} \cdot 4z$; last step is hiding expansion into events via $H(X|Y) = \sum p(y) H(X|Y=y)$ and filtering by $\geq 4z$ type-1 steps.

Formal theorem statements

Theorem 9.
Random tape δ-error adversarially robust algorithms for MIF (n, ℓ), with $\delta \leq \frac{\ell}{2^7 n}$, require space:

$$\Omega\left(\max_{k \in \mathbb{N}} \left(\frac{\ell^{k+1}}{n}\right)^{\frac{2}{k^2+3k-2}}\right) = \Omega\left(\ell^{\frac{15}{32}\log_n \ell}\right)$$

Theorem 10.
There is a family of adversarially robust random tape algorithms, where for MIF (n, ℓ) the corresponding algorithm has $\leq \delta$ error and uses

$$O\left(\left\lceil \frac{(4\ell)^{\frac{2}{d-1}}}{(n/4)^{\frac{3}{d(d-1)}}} \right\rceil (\log \ell)^2 + \min\left(\ell, \log\frac{1}{\delta}\right)\log \ell\right)$$

bits of space, where $d = \max\left(2, \min\left(\lceil \log \ell \rceil, \left\lfloor 2\frac{\log n/4}{\log(16\ell)} \right\rfloor\right)\right)$. When $\delta = 1/poly(n)$ a weakened space bound is $O\left(\ell^{\log_n \ell}(\log \ell)^2 + \log \ell\right)$.

Formal theorem statements

Theorem 11.
Pseudo-deterministic δ-error random oracle algorithms for MIF (n, ℓ) require

$$\Omega\left(\min\left(\frac{\ell}{\log\frac{2n}{\ell}} + \sqrt{\ell}, \frac{\ell\log\frac{1}{2\delta}}{\left(\log\frac{2n}{\ell}\right)^2\log n} + \left(\ell\log\frac{1}{2\delta}\right)^{1/4}\right)\right)$$

bits of space when $\delta \leq \frac{1}{3}$. In particular, when $\delta = 1/poly(n)$ and $\ell = \Omega(\log n)$, this is

$$\Omega\left(\frac{\ell}{\left(\log\frac{2n}{\ell}\right)^2} + (\ell\log n)^{1/4}\right)$$

Theorem 12.
Adversarially robust random seed algorithms for MIF (n, ℓ) with error $\leq \frac{1}{6}$ require space:

$$\Omega\left(\frac{\ell^2}{n} + \sqrt{\frac{\ell}{(\log n)^3}} + \ell^{1/5}\right)$$

Reduction step: searching for information must stop

1. Adversary sends $\ell/2$ random elements
   - Let ρ be resulting algorithm state
   - Not enough space to store all random elements: algorithm must overestimate, and only considers elements in a set H_ρ to be *safe*
   - Typically $|H_\rho| = O\left(\frac{zn}{\ell}\right)$
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   - Case (b): DONE – algorithm solves *MIF* $\left(O\left(\frac{zn}{\ell}\right), \Theta\left(\frac{\ell}{z}\right)\right)$ with inputs in W

Very simplified proof sketch

Generalization of deterministic lower bound from Stoeckl 2023

- For any state σ, integer q, let $FCO(\sigma, q)$ be set of "most common outputs" after q more inputs, with size w_q
- Interpret partial input stream $x \in [n]^\star$ as a state of the "canonical protocol"; then $FCO(x, q)$ gives most common canonical outputs
- We can recursively define FCO and hence "common outputs" so that we can prove:
  - If σ is a random state resulting from input x, then w.h.p. $FCO(\sigma, q) = FCO(x, q)$
  - $FCO(x, q) \cap x = \emptyset$
  - $|FCO(x, q)| \approx 2^{q/z}$, where the algorithm uses z bits of state
    - Pseudo-determinism used here: output built from dependent evaluations
- Since $n \geq |FCO(\epsilon, \ell)| \approx 2^{\ell/z}$, it follows $z \gtrapprox \frac{\ell}{\log n}$

FindCommonOutputs

- B is input to output function implemented by algorithm or canonical; $C \in_R [1,2)^{d \times N}$, x is stream prefix, and epochs are $t_d + \ldots + t_1 = \ell$; x has length $t_d + \ldots + t_{k+1}$. S is set of possible canonical outputs. $FCO(\cdots, k)$ output size is w_k, with all $w_k \geq \frac{5}{4} w_{k-1}$.

FindCommonOutputs(B, C, x, k)[9]

if $k = 1$

return iteratively extracted w_1 distinct elements, or error

$Q \leftarrow FCO(B, C, x \circ \langle 1, \ldots, t_k \rangle, k-1)$

for each $j \in S$

$f_j \leftarrow \left| \left\{ y \in \binom{Q}{t_k} : j \in FCO(B, C, x \circ \mathrm{sorted}(y), k-1) \right\} \right|$

$\theta \leftarrow C_{k,h} w_{k-1} / 16 |S|$

$P \leftarrow \left\{ j \in S : f_j^{(h)} \geq \theta \binom{|Q|}{t_k} \right\}$

return first w_k elements of $Q \cup P$

[9]This includes a simplification not present in published work.

Related work

- Generic methods to convert static to adversarial setting: Ben-Eliezer, Jayaram, Woodruff, and Yogev 2020; Ben-Eliezer and Yogev 2020, and recent diff. privacy approaches (which use random-tape) Ben-Eliezer, Eden, and Onak 2022; Hassidim, Kaplan, Mansour, Matias, and Stemmer 2020
- Static vs. adversarial separations: Assadi, Chen, and G. Sun 2022; Chakrabarti, Ghosh, and Stoeckl 2022; Kaplan, Mansour, Nissim, and Stemmer 2021
- White-box adversaries: Ajtai, Braverman, Jayram, Silwal, A. Sun, Woodruff, and Zhou 2022

- Read-once vs read-multiple use of randomness: Nisan 1993[10]
- Other work on Missing Item Finding and variants: Chakrabarti, Ghosh, and Stoeckl 2022; Magen 2024; Stoeckl 2023, 2024; Tarui 2007

[10] Noam Nisan. On read once vs. multiple access to randomness in logspace. *Theoretical Computer Science*, 107(1):135–144, 1993